

# $\mathcal{D}$ -modules with finite support are semi-simple

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## Abstract

Let  $(R, \mathfrak{m}, k_R)$  be regular local  $k$ -algebra satisfying the weak Jacobian criterion, such that  $k_R/k$  is an algebraic field extension. Let  $\mathcal{D}_R$  be the ring of  $k$ -linear differential operators of  $R$ . We give an explicit decomposition of the  $\mathcal{D}_R$ -module  $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}_R^{n+1}$  as a direct sum of simple modules, all isomorphic to  $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}$ , where certain ‘‘Pochhammer’’ differential operators are used to describe generators of the simple components.

## 1 Introduction

The reason for this note is that J.-E. Björk teased me by asking whether I knew if maximal ideals in a polynomial ring  $A = k[x_1, \dots, x_d]$  generate maximal left ideals in the Weyl algebra  $\mathcal{D}_A$ , also when the ground field  $k$  is not algebraically closed. That  $N = \mathcal{D}_A/\mathcal{D}_A\mathfrak{m}_A$  is semisimple follows from Kashiwara’s theorem that the category of  $\mathcal{D}_A$ -modules with support in the point  $\mathfrak{m}_A \in \text{Spec } A$  is equivalent to the category of vector space over the residue field  $k_A = A/\mathfrak{m}_A$ , where the equivalence is  $N \mapsto N^{\mathfrak{m}_A} = \{n \in N \mid \mathfrak{m} \cdot n = 0\}$  (see [1, V.3.1.2, VI.7.3]; the argument works fine also when  $k_A$  is not algebraically closed). In particular,  $N$  will be simple if  $\dim_{k_A} N^{\mathfrak{m}_A} = 1$ . We will prove this, which surely is well-known, but the main purpose of this note is to give explicit semi-simple decompositions of  $\mathcal{D}_R$ -modules with a finite support, where  $\mathcal{D}_R$  is the ring of differential operators associated to a rather general regular local ring  $R$ .

We shall work over a local regular noetherian  $k$ -algebra  $(R, \mathfrak{m}, k_R)$  of characteristic 0, only requiring that the  $R$ -module of  $k$ -linear derivations  $T_{R/k}$  is big enough.

**Theorem 1.1.** ([4, Thms. 30.6, 30.8]) *Let  $(R, \mathfrak{m}_R)$  be a regular local ring of dimension  $n$  containing the rational numbers  $\mathbf{Q}$ . Let  $R^*$  be a completion of  $R$ ,  $k_1$  a quasi-coefficient field of  $R$ , and  $K$  be a coefficient field of  $R^*$  such that  $k_1 \subset K$ . The following conditions are equivalent:*

- (1) *There exist  $\partial_1, \dots, \partial_n \in T_{R/k_1}$  and  $f_1, \dots, f_n \in \mathfrak{m}_R$  such that  $\det \partial_i(f_j) \notin \mathfrak{m}_R$ .*
- (2) *If  $\{x_1, \dots, x_n\}$  is a regular system of parameters and  $\partial_{x_i}$  are the partial derivatives of  $R^* = K[[x_1, \dots, x_n]]$ ,  $\partial_{x_i}(x_j) = \delta_{ij}$ , then  $\partial_{x_i} \in T_{R/k_1}$ .*

(3)  $T_{R/k_1}$  is free of rank  $n$ .

Furthermore, if these conditions hold, then for any  $P \in \text{Spec } R$ , putting  $A = R/P$ , we have  $T_{A/k_1} = T_{R/k_1}(P)/PT_R$ , and  $\text{rank } T_{A/k_1} = \dim A$ . ( $T_{R/k_1}(P) \subset T_{R/k_1}$  denotes the submodule of derivations  $\partial$  such that  $\partial(P) \subset P$ .)

If the equivalent conditions in Theorem 1.1 hold, then we say that  $(R, \mathfrak{m}, k_R)$  satisfies the weak Jacobian condition  $(WJ)_{k_1}$ . Note that if  $k_R/k$  is algebraic, then we can replace  $k_1$  by  $k$  and write  $(WJ)_k$ . In this paper  $(R, \mathfrak{m}, k_R)$  denotes a  $k$ -algebra of characteristic 0 satisfying  $(WJ)_k$ , where the field extension  $k_R/k$  is algebraic.

For example,  $R$  could be the localisation at a maximal ideal of a regular ring of finite type over  $k$ , a formal power series ring over  $k$ , or a ring of convergent power series when  $k$  is either the field of real or complex numbers.

Recall that the ring of ( $k$ -linear) differential operators  $\mathcal{D}_R \subset \text{End}_k(R)$  of  $R$  is defined inductively as  $\mathcal{D}_R = \cup_{m \geq 0} \mathcal{D}_R^m$ ,  $\mathcal{D}_R^0 = \text{End}_R(R) = R$ ,  $\mathcal{D}_R^{m+1} = \{P \in \text{End}_k(R) \mid [P, R] \subset \mathcal{D}_R^m\}$ , where  $[P, R] = PR - RP \subset \text{End}_k(R)$ . It is easy to see that  $T_R \subset \mathcal{D}_R^1 \subset \mathcal{D}_R$ , and conversely, if  $P \in \mathcal{D}_R^1$ , then  $P - P(1) \in T_R$ ; hence

$$\mathcal{D}_R^1 = R + T_R.$$

The following companion to Theorem 1.1 should be well known; see [3].

**Proposition 1.2.** *Let  $R/k$  be a regular local  $k$ -algebra satisfying  $(WJ)_k$  and such that  $k_R/k$  is algebraic. Then  $\mathcal{D}_R^1$  generates the algebra  $\mathcal{D}_R$ .*

Select  $x_i$  and  $\partial_{x_i}$  as in Theorem 1.1. Given a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $n = \dim R$ , we put  $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \in R$ ,  $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n} \in \mathcal{D}_R$ ,  $|\alpha| = \sum \alpha_i$ , and  $\alpha! = \alpha_1! \dots \alpha_n!$ .

We recall some important well-known facts for the algebra  $\mathcal{D}_R$ , which we later on will refer to as (Facts):

- (1) the  $R$ -module  $\mathcal{D}_R$  is free with basis  $\{\partial^\alpha\}_{\alpha \in \mathbf{N}^d}$ , where  $\mathcal{D}_R$  is either regarded as left or right module.
- (2)  $R$  is a simple  $\mathcal{D}_R$ -module.
- (3)  $\mathcal{D}_R$  is a simple ring.

*Proof.* (1): That the  $\partial^\alpha$  generate  $\mathcal{D}_R$  both as left or right  $R$ -algebra follows from Proposition 1.2. First consider  $\mathcal{D}_R$  as left  $R$ -module. Assume that  $P = \sum_{\alpha \in \Omega} a_\alpha \partial^\alpha = 0$ , where  $\Omega$  is a finite set of multi-indices. If one of the indices has minimal  $|\alpha|$  in the set  $\Omega$ , then  $P(X^\alpha) = a_\alpha \alpha! = 0$ . This implies that  $a_\alpha = 0$  for all  $\alpha$ . Now take the right module structure, and assume  $\sum_{\alpha \in \Omega} \partial^\alpha a_\alpha = 0$ . Then  $\sum_{\alpha \in \Omega} (a_\alpha \partial^\alpha + [\partial^\alpha, a_\alpha]) = 0$ , where  $[\partial^\alpha, a_\alpha] \in \mathcal{D}_R^{|\alpha|-1}$  and  $a_\alpha \partial^\alpha \in \mathcal{D}_R^{|\alpha|}$ . Since the  $\partial^\alpha$  are free generators as left module, it follows that if  $\alpha \in \Omega$  has maximal  $|\alpha|$ , then  $a_\alpha = 0$ . This implies that all  $a_\alpha = 0$ . (2): Let  $I \subset R$  be a non-zero  $\mathcal{D}_R$ -module. If  $I \neq R$  there exists a non-zero element  $f \in I \cap \mathfrak{m}^l$

of smallest  $l \geq 1$ . But then there exists a derivation  $\partial$  such that  $\partial(f) \in \mathfrak{m}^{l-1}$ ,  $\partial(f) \neq 0$ , which gives a contradiction. (3): If  $P \in \mathcal{D}_R^n$  belongs to a 2-sided ideal  $J$ , then  $P_r = [r, P] \in J \cap \mathcal{D}_R^{n-1}$  for all  $r \in R$ . Unless  $P \notin \mathcal{D}_R^0 = R$  there exists an element  $r$  such that  $P_r \neq 0$ . Iterating, it follows that  $J \cap R \neq 0$ . By (2)  $R \subset J$ ; hence  $J = \mathcal{D}_R$ .  $\square$

## 2 $\mathcal{D}$ -modules with finite support

Let  $\mathcal{D}_X = \mathcal{D}_{X/k}$  denote the sheaf of differential operators on a scheme  $X/k$ ; we refer to [2] for the basic definitions. Instead of schemes we could in a similar way consider sheaves on complex or real analytic manifolds (or even ringed spaces where the local rings are regular and satisfy  $(WJ)_k$  at all closed points), but the reader will have little problems in transcribing the theorem below to such a situation.

The theorem below can be regarded as a version of Kashiwara's embedding theorem:

**Theorem 2.1.** *Let  $X/k$  be a scheme of characteristic 0 such that the local ring at all closed points are regular and satisfies  $(WJ)_k$ , and that all closed points are rational over  $k$ . Let  $M$  be a coherent  $\mathcal{D}_{X/k}$ -module whose support  $\text{supp } M \subset X$  is a finite set of closed points. Let  $n_x$  be the length of the maximal submodule of  $M$  with support at the point  $x$ , and let  $\mathfrak{m}_x$  be the sheaf of ideals of  $x$ . Then  $M$  is a semi-simple module of the form*

$$M = \bigoplus_{x \in \text{supp } M} \bigoplus^{n_x} \frac{\mathcal{D}_X}{\mathcal{D}_X \mathfrak{m}_x},$$

and  $n_x = \dim_{k_x} M^{\mathfrak{m}_x}$ .

We remark that by (Fact 1)  $\mathcal{D}_R/\mathcal{D}_R \mathfrak{m} = k_R[\partial_{x_1}, \dots, \partial_{x_d}]$ , where the action of  $k_R[x_1, \dots, x_d]$  on the right side is determined by  $X^\alpha \partial^\beta = -\frac{\beta!}{\alpha!} \partial^{\beta-\alpha}$ , when  $\beta_i \geq \alpha_i$ ,  $i = 1, \dots, n$ , and otherwise  $X^\alpha \partial^\beta = 0$ .

Our goal is to give a concrete decomposition when we are given a presentation in terms of cyclic modules. First we have the following well-known lemma:

**Lemma 2.2.** ([5]) *Let  $\mathcal{D}$  be a simple ring (it has no non-trivial 2-sided ideals) and  $M$  be a  $\mathcal{D}$ -module of finite length. Assume that for any element  $m$  in  $M$  there exists a non-zero element  $P$  in  $\mathcal{D}$  such that  $Pm = 0$  ( $M$  is a torsion module). Then  $M$  is cyclic.*

We remark that artinian  $\mathcal{D}$ -modules are torsion in the above sense if the ring  $\mathcal{D}$  is not artinian. For example,  $\mathcal{D}_R$  is not artinian as soon as  $\dim R \geq 1$ , so in particular any  $\mathcal{D}_R$ -module of finite length is cyclic.

If now  $M$  is a  $\mathcal{D}_R$ -module of finite type with support at the maximal ideal  $\mathfrak{m}$ , then  $\dim_{k_R} M^{\mathfrak{m}} < \infty$ , and any set of generators of  $M$  will belong to  $M^{\mathfrak{m}^n}$  for sufficiently high  $n$ . Therefore  $M$  cannot have an infinite composition series, i.e.  $M$  is of finite length, and since any element in  $M$  is killed by  $\mathfrak{m}^n$  for sufficiently

high  $n$ , it is clearly a torsion module (which we thus can see without using the fact that  $\mathcal{D}_R$  is non-artinian); hence  $M$  is cyclic by Lemma 2.2. If  $m$  is a cyclic generator we have a surjective homomorphism  $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1} \rightarrow \mathcal{D}_R m = M$ , so that after iteration we have a finite resolution

$$0 \rightarrow \frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n_r}} \rightarrow \cdots \rightarrow \frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n_i}} \rightarrow \cdots \rightarrow \frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n+1}} \rightarrow M \rightarrow 0. \quad (2.1)$$

*Proof of Lemma 2.2.* We prove this by induction over the length of a  $\mathcal{D}$ -module  $M$ . If  $l(M) = 1$ ,  $M$  is simple so any non-zero vector is a cyclic generator. Now assume  $l(M) \geq 2$  and that the assertion holds for all modules of length  $< l(M)$ . If  $L \subset M$  is a non-zero simple submodule, we have the exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow M/L \rightarrow 0$$

where  $l(M/L) < l(M)$ . By assumption there exists an element  $m$  in  $M$  that maps to a cyclic generator in  $M/L$ . Choose a non-zero vector  $m_0 \in L$ . Since  $M$  is a torsion module,  $\text{Ann}_{\mathcal{D}}(m) \neq 0$ , and since  $\mathcal{D}$  is simple, the 2-sided ideal  $\text{Ann}_{\mathcal{D}}(m)\mathcal{D}$  contains the identity 1; hence there exists  $Q \in \text{Ann}_{\mathcal{D}}(m)$  and  $P \in \mathcal{D}$  such that  $Q P m_0 \neq 0$ . Putting  $m_1 = m + P m_0$ , we have  $Q m_1 = Q P m_0 \in L$ , and since  $L$  is simple, both  $P m_0$  and  $m_0$  belong to  $\mathcal{D} m_1$ ; hence also  $m \in \mathcal{D} m_1$ . By assumption any element  $m'$  in  $M$  can be written  $m' = P_0 m_0 + P_1 m$ ; since  $m, m_0 \in \mathcal{D} m_1$  this shows that  $m_1$  is a cyclic generator of  $M$ .  $\square$

Recall the Pochhammer symbol:

$$(a)_n = a(a+1) \cdots (a+n-1),$$

and we also put  $(a)_0 = 1$ . In the theorem below we use the notation in Theorem 1.1.

**Theorem 2.3.** *Let  $(R, \mathfrak{m}, k_R)$  be an allowed regular local  $k$ -algebra  $R$  of dimension  $d$ , such that the residue field  $k_R = R/\mathfrak{m}$  is algebraic over  $k$ , and let  $\mathcal{D}_R$  be the ring of differential operators of  $R$ . Define the derivations  $\partial_{x_i}$  by  $\partial_{x_i}(x_j) = \delta_{ij}$  and the ‘‘Pochhammer’’ differential operators*

$$Q_{n,d}(x_1, \dots, x_d) = \prod_{i=1}^d (1 + \partial_{x_i} x_i)_n \in \mathcal{D}_R.$$

(1)  $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}$  is a simple  $\mathcal{D}_R$ -module and  $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1}$  is a semi-simple  $\mathcal{D}_R$ -module for each positive integer  $n$ .

(2) There is an isomorphism of  $\mathcal{D}_R$ -modules

$$\begin{aligned} \psi : \bigoplus_{j=0}^n \bigoplus_{|\alpha|=j} \frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}} &\rightarrow \frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n+1}}, \\ (P_{\alpha,j} \bmod \mathcal{D}_R\mathfrak{m}) &\mapsto P_{\alpha,j} Q_{n-j,d}(x_1, \dots, x_d) X^\alpha \bmod \mathcal{D}_R\mathfrak{m}^{n+1} \end{aligned}$$

**Lemma 2.4.** *Let  $M$  be a  $\mathcal{D}_R$ -module which is generated by its  $\mathfrak{m}$ -invariant subspace  $M^{\mathfrak{m}} = \{m \in M \mid \mathfrak{m} \cdot m = 0\}$ . Then  $M$  is semi-simple. More precisely, if  $S$  is a basis of the  $k_R$ -vector space  $M^{\mathfrak{m}}$ , we have*

$$M = \bigoplus_{v \in S} \mathcal{D}_R v,$$

where all the modules  $\mathcal{D}_R v$  are isomorphic to the simple module  $\mathcal{D}_R/\mathcal{D}_R \mathfrak{m}$ .

*Proof.* We first note that if  $L$  is a  $\mathcal{D}_R$ -module of finite type which is generated by the invariant space  $L^{\mathfrak{m}}$ , and this space is one-dimensional over  $k_R$ , then  $L$  is simple. This follows since any element in  $L$  is killed by a sufficiently high power of  $\mathfrak{m}$ , so if  $L_1$  is a non-zero submodule we have  $L_1^{\mathfrak{m}} \neq 0$ . Hence  $L_1^{\mathfrak{m}} = L^{\mathfrak{m}}$ , which gives  $L_1 = L$ .

To see that the module  $N = \mathcal{D}_R/\mathcal{D}_R \mathfrak{m}$  is simple, by the previous paragraph it suffices to prove that  $N^{\mathfrak{m}}$  is one-dimensional over  $k_R$ . So if  $P \in \mathcal{D}_R$  and  $P \bmod \mathcal{D}_R \mathfrak{m} \in N^{\mathfrak{m}}$ , i.e.  $\mathfrak{m}P \subset \mathcal{D}_R \mathfrak{m}$ , we need to see that  $P \in R + \mathcal{D}_R \mathfrak{m}$ . Expressed in a regular system of parameters  $P = \sum_{\alpha} \partial^{\alpha} a_{\alpha}$  we have  $(x_1, \dots, x_d) \cdot \sum_{\alpha} \partial^{\alpha} a_{\alpha} \subset \mathcal{D}_R \cdot (x_1, \dots, x_d)$ . This implies, from the fact that the differential operators  $\partial^{\alpha}$  form a basis of the right  $R$ -module  $\mathcal{D}_R$ , that  $a_{\alpha} \in (x_1, \dots, x_d)$  when  $|\alpha| > 0$ , and therefore  $P \in R + \mathcal{D}_R \mathfrak{m}$ .

If  $v \in M^{\mathfrak{m}}$ , then there is a canonical non-zero homomorphism  $\mathcal{D}_R/\mathcal{D}_R \mathfrak{m} \rightarrow \mathcal{D}_R v$ , which is injective and has a simple image by the previous paragraph. The canonical surjective homomorphism

$$\bigoplus_{v \in S} \mathcal{D}_R v \rightarrow M$$

is an isomorphism since the left hand side is semi-simple and the restriction to any of its simple terms is non-zero.  $\square$

*Proof of Theorem 2.1 and Theorem 2.3.* The decomposition of  $M$  over the support is obvious, so one can assume that  $\text{supp } M = \{x\}$  is a single closed point in  $X$ , and thus  $M$  can be regarded as a  $\mathcal{D}_R$ -module, where  $R$  is the local ring  $\mathcal{O}_{X,x}$ . Let  $M^0$  be an  $R$ -submodule of finite type that generates  $M$ , hence  $\mathfrak{m}^{n+1}M^0 = 0$  for high  $n$ ; let  $n$  be the highest integer such that  $\mathfrak{m}^n M^0 \neq 0$ . Put  $M_i^0 = \mathfrak{m}^i M^0$  and let  $M_i$  be the  $\mathcal{D}_R$ -module it generates, so we have a filtration by  $\mathcal{D}_R$ -modules  $M_n \subset M_{n-1} \subset \dots \subset M_0 = M$ , and exact sequences

$$0 \rightarrow M_{i+1} \rightarrow M_i \rightarrow M_i/M_{i+1} \rightarrow 0,$$

Then  $M_n$  and each quotient  $M_i/M_{i+1}$  is generated by its  $\mathfrak{m}$ -invariants, so at any rate  $M$  is a successive extension of  $\mathcal{D}_R$ -modules that are generated by  $\mathfrak{m}$ -invariants. All these modules can be decomposed into a direct sum of simple modules that are isomorphic to the module  $\mathcal{D}_R/\mathcal{D}_R \mathfrak{m}$ . It remains to see that  $M$  is a direct sum of such modules. To see this first note that  $M$  is a quotient of a direct sum of modules of the form  $\mathcal{D}_R/\mathcal{D}_R \mathfrak{m}^{n+1}$  for different non-negative integers  $n$ , so to see that  $M$  is semi-simple it suffices to see that  $\mathcal{D}_R/\mathcal{D}_R \mathfrak{m}^{n+1}$  is semi-simple, and this follows if we prove (2) in Theorem 2.3.

We have  $x_i^k(k + \partial_{x_i} x_i) = \partial_{x_i} x_i^{k+1}$ , hence  $x_i(1 + \partial_{x_i} x_i)_{n-j} = \partial_{x_i}^{n-j} x_i^{n+1-j}$ . Therefore  $x_i Q_{n-j,d}(x_1, \dots, x_d) X^\alpha = Q_{n-j,d-1}(x_1, \dots, \hat{x}_i, \dots, x_d) x_i^{n+1-j} X^\alpha$ , so if  $|\alpha| = j$ , then

$$\mathfrak{m} Q_{n-j,d}(x_1, \dots, x_d) X^\alpha \subset \mathcal{D}_R \mathfrak{m}^{n+1}. \quad (*)$$

Therefore there exists a homomorphism of  $\mathcal{D}_R$ -modules

$$\begin{aligned} \psi_\alpha : \frac{\mathcal{D}_R}{\mathcal{D}_R \mathfrak{m}} &\rightarrow \frac{\mathcal{D}_R}{\mathcal{D}_R \mathfrak{m}^{n+1}} \\ P &\mapsto P Q_{n-j,d}(x_1, \dots, x_d) X^\alpha \bmod \mathcal{D}_R \mathfrak{m}^{n+1}, \end{aligned}$$

and we put  $\psi_\alpha(1_\alpha) = m_\alpha \in (\frac{\mathcal{D}_R}{\mathcal{D}_R \mathfrak{m}^{n+1}})^\mathfrak{m}$ , where  $1_\alpha$  is the cyclic generator of the term with index  $\alpha$  in the right side of (2).

The fact that any differential operator  $P \in \mathcal{D}_R$  has a unique expansion  $P = \sum \partial^\alpha a_\alpha$ ,  $a_\alpha \in R$ , implies that  $Q_{n-j,d}(x_1, \dots, x_d) X^\alpha \notin \mathcal{D}_R \mathfrak{m}^{n+1}$ , when  $|\alpha| \leq j$ ; hence  $\psi_\alpha \neq 0$ .

Lemma 2.4 implies that  $\psi$  is injective if we first prove that the vectors  $m_\alpha$  are linearly independent in the  $k_R$ -vector space  $(\frac{\mathcal{D}_R}{\mathcal{D}_R \mathfrak{m}^{n+1}})^\mathfrak{m}$ . Assume that we have a linear relation

$$\sum_{|\alpha| \leq n} \lambda_\alpha m_\alpha = 0, \quad \lambda_\alpha \in k_A,$$

which means

$$\sum_{|\alpha| \leq n} \hat{\lambda}_\alpha Q_{n-j} X^\alpha \in \mathcal{D}_R \mathfrak{m}^{n+1}, \quad \hat{\lambda}_\alpha \in R, \quad \hat{\lambda}_\alpha \bmod \mathfrak{m} = \lambda_\alpha.$$

Defining the Euler operator  $\nabla = \sum_{i=1}^d x_i \partial_{x_i}$  we have  $[\nabla, Q_{n-j}] = 0$ ,  $[\nabla, X^\alpha] = |\alpha| X^\alpha$ , and

$$\hat{\lambda}_\alpha Q_{n-j} X^\alpha \nabla = (d - |\alpha|) \hat{\lambda}_\alpha Q_{n-j} X^\alpha - \nabla(\hat{\lambda}_\alpha) Q_{n-j} X^\alpha + (\nabla - d) \hat{\lambda}_\alpha Q_{n-j} X^\alpha.$$

Here the two last terms on the right belong to  $\mathcal{D}_R \mathfrak{m}^{n+1}$  due to (\*), after noting that  $\nabla(\hat{\lambda}_\alpha) \in \mathfrak{m}$  and  $\nabla - d = \sum_{i=1}^d \partial_{x_i} x_i$ . Therefore we can define a  $k_R$ -linear action  $E$  on the linear space  $\sum_{|\alpha| \leq n} k_R m_\alpha \subset (\frac{\mathcal{D}_R}{\mathcal{D}_R \mathfrak{m}^{n+1}})^\mathfrak{m}$  such that  $E m_\alpha = (d - |\alpha|) m_\alpha$ . A standard weight argument now implies that all the coefficients  $\lambda_\alpha = 0$ .

It remains to prove that  $\psi$  is an isomorphism. Let  $N_n \subset \mathcal{D}_R / \mathcal{D}_R \mathfrak{m}^{n+1}$  be the submodule that is generated by the canonical projection of  $\mathfrak{m}^n$  in  $\mathcal{D}_R / \mathcal{D}_R \mathfrak{m}^{n+1}$ . Then  $N_n$  is generated by its  $\mathfrak{m}$ -invariants and  $\dim_{k_R} N_n^\mathfrak{m}$  equals the number of monomials of degree  $n$  in  $d$  variables, which is thus equal to the length of  $N_n$  by Lemma 2.4. Since  $\mathcal{D}_R / \mathcal{D}_R \mathfrak{m}^{n+1} / N_n = \mathcal{D}_R / \mathcal{D}_R \mathfrak{m}^n$ , an induction over  $n$  gives that the lengths of both sides in (2) are equal. Since  $\psi$  is injective this implies that  $\psi$  is an isomorphism.  $\square$

**Remark 2.5.** (1) The proof gives

$$(\frac{\mathcal{D}_R}{\mathcal{D}_R \mathfrak{m}^{n+1}})^\mathfrak{m} = \sum_{|\alpha| \leq n} k_R Q_{n-j,d}(x_1, \dots, x_d) X^\alpha \bmod \mathcal{D}_R \mathfrak{m}^{n+1}.$$

- (2) Given a resolution as in (2.1), then (2) in Theorem 2.3 will give a decomposition of  $M$ .

**Example 2.6.** Let  $A_1(k)$  be the Weyl algebra in one variable over a field  $k$  of characteristic 0, and consider a maximal ideal  $\mathfrak{m}$  in the polynomial ring  $k[x] \subset A_1(k)$ ; let  $\partial_x$  be the  $k$ -linear derivation of  $k[x]$  such that  $\partial_x(x) = 1$ . By Theorem 2.1,  $M_l = A_1(k)/A_1(k)\mathfrak{m}^{l+1} = A_1(k)m_l$  is a semi-simple  $A_1(k)$ -module (here  $m_l = 1 \bmod A_1(k)\mathfrak{m}^{l+1}$ ). The localisation  $R = k[x]_{\mathfrak{m}}$  has a regular system of parameters formed by a generator  $x_1$  of the principal ideal  $\mathfrak{m}$ , and given  $x_1$  there exists a unique derivation  $\partial_{x_1} \in T_{R/k}$  such that  $\partial_{x_1}(x_1) = 1$ . The  $\mathcal{D}_{R/k}$ -module  $R \otimes_{k[x]} M_l$  can be decomposed as follows:

$$\bigoplus_{i=0}^l R \otimes_{k[x]} M_0 \rightarrow R \otimes_{k[x]} M_l,$$

$$\begin{aligned} (P_i m_{0,i}) &\mapsto P_0 \cdot x_1^l m_l + P_1 \cdot (1 + \partial_{x_1} \cdot x_1) x_1^{l-1} m_l + \cdots \\ &+ P_l \cdot (1 + \partial_{x_1} \cdot x_1)(2 + \partial_{x_1} \cdot x_1) \cdots (l + \partial_{x_1} \cdot x_1) m_l. \end{aligned}$$

The isomorphism  $M_0 \oplus \cdots \oplus M_0 \rightarrow M_l$ , where there are  $l+1$  terms on the left, is defined similarly. We notice that although  $\partial_{x_1} x_1$  in general does not act on  $k[x]$ , it has a well-defined action on  $M_l$ . Note also that the invariant space  $M_0^{\mathfrak{m}}$  of the simple module  $M_0$  is 1-dimensional over the residue field  $k_R$  and thus its dimension over  $k$  equals the degree of the field extension  $k_R/k$ .

One can reverse the roles of  $\partial_x$  and  $x$  in the Weyl algebra  $A_1(k)$ , and instead decompose modules according to their support in  $\text{Spec } k[\partial_x]$ . Thus if  $P \in k[\partial_x] \subset A_1(k)$  is a differential operator with constant coefficients, then

$$\frac{A_1(k)}{A_1(k)P} \cong \frac{A_1(k)}{A_1(k)P_1} \oplus \cdots \oplus \frac{A_1(k)}{A_1(k)P_r}$$

where  $P = P_1 \cdots P_r$  is a factorisation into irreducible polynomials, where repetitions may occur. It is a good exercise to write down an isomorphism for some concrete polynomial  $P$  using ‘‘Pochhammer’’ operators  $Q_{j,1}(P_i)$  when  $P$  has multiple factors.

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